

# The Vacuum in Nonisentropic Gas Dynamics

Geng Chen\*, Robin Young†

November 29, 2011

## Abstract

We investigate the vacuum in nonisentropic gas dynamics in one space variable, with the most general equation of states allowed by thermodynamics. We recall physical constraints on the equations of state and give explicit and easily checkable conditions under which vacuums occur in the solution of the Riemann problem. We then present a class of models for which the Riemann problem admits unique global solutions without vacuums.

2000 *Mathematical Subject Classification*: 35L65, 35B65, 35B35.

*Key Words*: Nonisentropic gas dynamics, conservation laws, vacuum, Large data, Riemann problem.

## 1 Introduction

We consider the Euler equations of fluid dynamics in one space dimension,

$$\begin{aligned}\rho_t + (\rho u)_x &= 0, \\ (\rho u)_t + (\rho u^2 + p)_x &= 0, \\ (\tfrac{1}{2}\rho u^2 + \rho E)_t + (u(\tfrac{1}{2}\rho u^2 + \rho E + p))_x &= 0,\end{aligned}\tag{1.1}$$

describing conservation of mass, momentum and energy, respectively. Here  $\rho$  is the density,  $p$  is the pressure,  $u$  is the velocity, and  $E$  is the specific internal energy of the fluid: the equations are closed by specifying an equation of

---

\*chen@math.psu.edu. Department of Mathematics, Pennsylvania State University, University Park, PA, 16802

†young@math.umass.edu. Department of Mathematics, University of Massachusetts, Amherst, MA 01003. Supported in part by NSF Applied Mathematics Grant Number DMS-0908190.

state or constitutive relation, describing how the thermodynamic variables are related. This equation of state depends on the molecular structure of the fluid and is subject to physical constraints such as the Second Law of Thermodynamics.

The Cauchy problem for (1.1) is well understood when the initial data have small total variation [8, 1, 16, 7], but little is known for solutions with large data [18, 19]. One of the main difficulties is the possible occurrence of vacuum. Several authors have studied the vacuum in isentropic gas dynamics (obtained by dropping the energy equation) [10, 12, 15, 20], and more recently nonisentropic polytropic ideal ( $\gamma$ -law) gases [3, 4, 5].

Our goal here is to describe in detail which gases admit vacuums in the solution. It is well known that  $\gamma$ -law gases require vacuum in order to solve the global Riemann problem [16], whereas an isothermal gas ( $\gamma = 1$ ) does not. Also, a better existence theory is available for the isothermal system, due to a degenerate wave curve structure [14], but there is no natural  $3 \times 3$  physically consistent analogue of the isothermal system. In [17], Temple considers a class of  $3 \times 3$  constitutive relations with simplified structure, but this system violates some thermodynamic constraints, given in [11, 13]. Our intention is to present a class of physically consistent constitutive laws which do not admit vacuum, thus removing the issue of vacuum for these equations of state, in order to focus more fully on the effects of nonlinear wave interactions.

We begin by collecting all the physical conditions that restrict the equation of state. We recall the solution of the Riemann problem and give a necessary and sufficient condition for the occurrence of vacuums in the general solution of the Riemann problem. By exhibiting specific examples, we describe a class of constitutive laws which satisfy all our physical constraints and for which the Riemann problem does not contain a vacuum state. We then present the simplest class of such equations of state, namely

$$E = \frac{K_0 e^{S/c_\tau}}{2\sigma - 1} (\ln(\tau + 1))^{1-2\sigma}, \quad p = K_0 e^{S/c_\tau} \frac{(\ln(\tau + 1))^{-2\sigma}}{\tau + 1},$$

where  $\frac{1}{2} < \sigma \leq 1$ . Although it is implied by quoted results, we explicitly solve the global Riemann problem for this pressure law, without use of the vacuum state. It is expected that this will inform such a study of wave interactions, which is the subject of the authors' ongoing research.

## 2 Thermodynamic constraints

To set notation, we describe the thermodynamic constraints. The thermodynamic properties of a fluid are embodied in the constitutive relation  $E = E(\tau, S)$ , where  $\tau = 1/\rho$  is the specific volume and  $S$  is the specific entropy. See [13] for a physical discussion of these constraints, and [15] for a detailed mathematical analysis.

First, we make the *smoothness assumption*,

$$E = E(\tau, S) \in C^2, \quad \text{for } \tau \in \mathbb{R}^+ \quad \text{and} \quad S \in \mathbb{R}, \quad (2.1)$$

which is true for most fluids.

We require the fluid to satisfy the Second Law of Thermodynamics, which asserts that

$$dE = T dS - p d\tau, \quad (2.2)$$

where  $T$  is the temperature. This in turn implies that

$$E_S = T, \quad \text{and} \quad E_\tau = -p. \quad (2.3)$$

We assume the *standard thermodynamic constraints*: specific volume  $\tau$ , pressure  $p$  and temperature  $T$  satisfy

$$\tau > 0, \quad p > 0 \quad \text{and} \quad T \geq 0;$$

so that by (2.3),

$$E_\tau < 0 \quad \text{and} \quad E_S \geq 0.$$

We assume “stability of matter”, which asserts that the energy is finite,

$$E_\infty = \lim_{\tau \rightarrow \infty} E < \infty,$$

and without loss of generality we take

$$E_\infty = 0, \quad \text{so that} \quad E(\tau, S) > 0 \quad (2.4)$$

for all  $\tau > 0$  and  $S$ , see [11].

Next, we assume the *thermodynamic stability constraint* that the energy  $E = E(\tau, S)$  be jointly convex,

$$E_{\tau\tau} = -p_\tau > 0, \quad E_{SS} = T_S \geq 0, \quad (2.5)$$

while also

$$E_{\tau\tau} \cdot E_{SS} \geq E_{\tau S}^2 \quad \text{and} \quad E_{\tau S} = -p_S \leq 0. \quad (2.6)$$

According to [13], thermodynamic stability yields (2.5) and (2.6), but our discussion requires only (2.5). Equation (2.5) in turn implies that the system is strictly hyperbolic away from vacuum. The condition  $p_S > 0$  states that the material expands upon heating at constant pressure. We assume nonstrict inequality for  $p_S$  to include isentropic gas dynamics, for which  $p_S \equiv 0$ .

Our final condition is an **energy condition**, which states that if the pressure  $p(\tau, S)$  is specified, then the energy  $E$  is well defined: that is,

$$E(\tau, S) = \int_{\tau}^{\infty} p(\tau', S) d\tau',$$

where we have used (2.3) and (2.4). That is, we require that, for all  $S$ ,

$$\int_1^{\infty} p(\tau, S) d\tau = \int_0^1 \frac{\hat{p}(\rho, S)}{\rho^2} d\rho < +\infty, \quad (2.7)$$

where  $\hat{p}$  is defined by

$$\hat{p}(\rho, S) \equiv p(1/\rho, S) = p(\tau, S). \quad (2.8)$$

The energy condition (2.7) imposes growth conditions on  $p(\tau, S)$ , or equivalently restricts the pressure  $\hat{p}$  near vacuum, namely  $\hat{p}(0+, S) = 0$ , and by l'Hospital's rule,

$$\lim_{\rho \rightarrow 0} \hat{p}_\rho(\rho, S) = \lim_{\rho \rightarrow 0} \frac{\hat{p}(\rho, S)}{\rho} = 0. \quad (2.9)$$

Note that our conditions alone are *not* sufficient to conclude uniqueness of solutions to the Riemann problem: uniqueness is assured if and only if Smith's *medium condition*, that is

$$\frac{\partial}{\partial \tau} p(\tau, E) \leq \frac{p^2}{2E}, \quad (2.10)$$

where  $p$  is regarded as  $p(\tau, E)$ , is satisfied, see [15].

### 3 Vacuum in the solution of Riemann problems

We wish to investigate the circumstances in which a vacuum appears in the solution to a Riemann problem. We briefly recall the solution of the Riemann problem; see [16].

### 3.1 Riemann problem

We begin by calculating the simple (rarefaction) wave curves. For smooth solutions, we replace the third (energy) equation of (1.1) by the entropy equation

$$S_t + uS_x = 0,$$

and use  $(\rho, u, S)$  as the state variables. It is routine to calculate the eigen-system after writing (1.1) in quasilinear form. The eigenvalues of (1.1) are

$$\lambda_1 = u - c, \quad \lambda_2 = u, \quad \lambda_3 = u + c,$$

and these are the wavespeeds of the backward, middle and forward waves, respectively, and

$$c = c(\rho, S) := \sqrt{\hat{p}_\rho} \quad (3.1)$$

is the speed of sound. As is well known, the forward and backward waves are genuinely nonlinear and the middle waves linearly degenerate. The corresponding eigenvectors are

$$r_1 = \begin{pmatrix} \rho \\ -c \\ 0 \end{pmatrix}, \quad r_2 = \begin{pmatrix} \hat{p}_S \\ 0 \\ -\hat{p}_\rho \end{pmatrix}, \quad r_3 = \begin{pmatrix} \rho \\ c \\ 0 \end{pmatrix}.$$

It follows that the equation of a backward simple wave is

$$u - u_l = R(\rho_l, S) - R(\rho, S), \quad S = S_l, \quad (3.2)$$

where the subscript  $l$  refers to the left state of the wave, and we define

$$R(\rho, S) = \int_1^\rho \frac{c(r, S)}{r} dr = \int_1^\rho \frac{\sqrt{\hat{p}_\rho}}{r} dr, \quad \text{for } \rho \geq 0. \quad (3.3)$$

The equation of a forward simple wave curve is

$$u_r - u = R(\rho_r, S) - R(\rho, S), \quad S = S_r, \quad (3.4)$$

where the subscript  $r$  refers to the right state of the wave.

Next we calculate the shock curves: these are described by the Rankine-Hugoniot conditions,

$$\begin{aligned} \xi[\rho] &= [\rho u], \\ \xi[\rho u] &= [\rho u^2 + p], \\ \xi[\tfrac{1}{2}\rho u^2 + \rho E] &= [u(\tfrac{1}{2}\rho u^2 + \rho E + p)], \end{aligned} \quad (3.5)$$

where  $\xi$  is the shock speed and the brackets denote the jump in a quantity across the shock. We simplify these as follows: the first equation can be written

$$\xi \frac{\rho_r}{\rho_l} - \xi - \frac{\rho_r}{\rho_l} u_r + u_l = 0, \quad (3.6)$$

and, recalling that  $\tau = 1/\rho$ , using this in the second equation and simplifying yields

$$[p][\tau] = \xi[\rho u][\tau] - [\rho u^2][\tau] = -[u]^2.$$

Next, denoting the average of a quantity by  $\bar{g} = \frac{g_l + g_r}{2}$ , manipulating the first two equations of (3.5) yields

$$\xi \bar{p}[u] = \bar{\rho u}[u] + [p], \quad \text{so} \quad \xi \bar{p} - \bar{\rho u} = \frac{[p]}{[u]}. \quad (3.7)$$

The third equation of (3.5) gives, after simplifying,

$$\xi \bar{p}[\frac{1}{2}u^2 + E] = \bar{\rho u}[\frac{1}{2}u^2 + E] + [up].$$

Using (3.7) and again simplifying, we finally obtain

$$[E] + \bar{p}[\tau] = 0, \quad (3.8)$$

which is the *Hugoniot curve* for shocks. We conclude that this describes the shock curve fully: first, solve (3.8) to find the relation between  $\rho$  and  $S$ , then use

$$[u] = -\sqrt{-[p][\tau]}, \quad (3.9)$$

obtained from the entropy condition [9, 6], to resolve  $u$ , and finally use (3.7) to determine  $\xi$ .

Recall that an entropy condition is required to choose admissible shocks and thus obtain uniqueness of Riemann solutions [6, 9]. This condition states that pressure (and thus also density) is bigger behind the shock, and leads to the negative square root in (3.9) above. It follows similarly that the density behind a (forward or backward) rarefaction wave is *smaller* than the density ahead of the wave. Also, this implies that the state behind a shock *cannot* be the vacuum state.

It is routine to describe contact discontinuities using (3.5): namely, substitute  $[u] = 0$  in directly, to obtain

$$[u] = 0, \quad [p] = 0 \quad \text{and} \quad \xi = \bar{u} = u. \quad (3.10)$$

If a contact discontinuity is adjacent to a vacuum, then we combine the contact discontinuity and the vacuum region into a new vacuum region, called a *non-isentropic vacuum* on which the entropy density  $\rho S$  vanishes. The left and right hand limits of  $S$  on the left and right boundaries of such a vacuum region are different. It follows that, if the vacuum is involved in the solution of the Riemann problem, it can only be generated between two outgoing rarefaction waves, see [12, 10, 20]

### 3.2 Vacuum condition

**Lemma 3.1.** *The vacuum state exists in the solution of Riemann problems of (1.1) if and only if, for some  $S$ ,*

$$R(0+, S) > -\infty, \quad (3.11)$$

where  $R$  is defined in (3.3).

*Proof.* We first prove that if the vacuum state exists in the solutions of Riemann problems then (3.11) is satisfied. Recall that vacuum state only appears between two rarefaction waves. If a Riemann solution consists of forward and backward rarefactions, it follows that the velocity  $u$  is monotone increasing as a function of  $x$  [6]. Parameterizing the forward wave by  $\rho$ , it follows from (3.4) that

$$u_r - u = R(\rho_r, S_r) - R(\rho, S_r);$$

now since  $u \geq u_l$ , we get the uniform bound

$$R(\rho, S_r) \geq u_l - u_r + R(\rho_r, S_r),$$

and allowing  $\rho \rightarrow 0$  implies (3.11).

Now suppose (3.11) holds for some  $S$ . We claim that the Riemann problem with data  $U_l = (1, 0, S)$ , and  $U_r = (1, u_r, S)$  has a vacuum in the solution whenever

$$u_r > -2R(0+, S).$$

To see this, assume  $u_r > 0$  and resolve the Riemann problem into backward and forward rarefactions using (3.2) and (3.4), to get

$$\begin{aligned} u_m &= R(1, S) - R(\rho_m, S) = -R(\rho_m, S) \quad \text{and} \\ u_r - u_m &= R(1, S) - R(\rho_m, S) = -R(\rho_m, S), \end{aligned}$$

with no contact as  $S_r = S_l = S$ . Adding, we must solve

$$u_r = -2 R(\rho_m, S),$$

and so if  $u_r > -2 R(0+, S)$ , no such  $\rho_m$  can be found and a vacuum is required to solve the Riemann problem; see also [16, 20].  $\square$

We now introduce an easily checkable condition which implies (3.11), so is a sufficient condition for existence of Riemann solutions with vacuum. This **pressure near vacuum condition** describes the rate at which  $\hat{p} \rightarrow 0$ : for some value of  $S$ , there exist positive numbers  $\varepsilon_0$ ,  $\alpha_0$  and  $M_0$ , such that,

$$\hat{p}_\rho(\rho, S) \leq M_0 \rho^{\alpha_0} \quad \text{whenever} \quad \rho \in (0, \varepsilon_0). \quad (3.12)$$

Note that we require this condition at only one  $S$ : by continuity, we would generally expect the condition to hold in an open set of  $S$  values. Note also that polytropic ideal gases satisfy (3.12),  $\alpha_0$  being given by the adiabatic exponent  $\gamma > 0$ .

**Theorem 3.2.** *The energy condition (2.7) and pressure near vacuum condition (3.12) together imply that vacuums exist in the solution of some Riemann problems.*

*Proof.* According to (2.9), the energy condition implies that  $\hat{p}_\rho \rightarrow 0$  as  $\rho \rightarrow 0$ , so (3.12) makes sense. It then suffices by Lemma 3.1 to show that, for  $S$  given by (3.12), equation (3.11) is satisfied. From (3.3), for  $\rho < \varepsilon_0$ , we have

$$\begin{aligned} R(\varepsilon_0, S) - R(\rho, S) &= \int_\rho^{\varepsilon_0} \frac{\sqrt{\hat{p}_\rho(r, S)}}{r} dr \\ &\leq \int_\rho^{\varepsilon_0} \sqrt{M_0} r^{-1+\alpha_0/2} dr \\ &\leq \sqrt{M_0} \frac{2}{\alpha_0} \varepsilon_0^{\alpha_0/2}, \end{aligned}$$

and taking the limit  $\rho \rightarrow 0$  gives the required lower bound for  $R(0+, S)$ .  $\square$

## 4 Gas dynamics without vacuum

We now write down a class of constitutive laws for gases which *do not* admit vacuums in the solution of the Riemann problem. These gases satisfy all the constraints of Section 2, but do not satisfy the vacuum condition (3.11).



By (3.12), there is a vacuum for any  $\gamma$ -law gas ( $p \sim \rho^\gamma$ ) with  $\gamma > 1$ , but no vacuum for an isothermal gas ( $\gamma = 1$ ). We thus look for pressure laws between these cases, which restricts our equation of state.

It is convenient to work with separable energies, that is, energies of the form  $E(\rho, S) = f(S)g(\rho)$ . Specifically, we consider energies given by the expression

$$E(\tau, S) = f(S) \int_{\ln(\tau+1)}^{+\infty} k^2(y) dy, \quad (4.1)$$

where  $f$  and  $k$  are  $C^2$  functions satisfying certain conditions described below.

We assume that  $f(S)$  is  $C^2$ , positive, increasing and convex,

$$f(S) > 0, \quad f'(S) > 0, \quad f''(S) \geq 0, \quad (4.2)$$

with limits

$$\lim_{S \rightarrow \infty} f(S) = \infty \quad \text{and} \quad \lim_{S \rightarrow -\infty} f(S) = 0. \quad (4.3)$$

We assume  $k(y)$  is a  $C^2$ , convex, decreasing function defined on  $(0, \infty)$ ,

$$k(y) > 0, \quad k'(y) < 0, \quad k''(y) > 0, \quad (4.4)$$

subject to the growth limits

$$\int_1^{+\infty} k^2(y) dy < +\infty \quad \text{and} \quad \int_1^{+\infty} k(y) dy = +\infty, \quad (4.5)$$

describing growth near vacuum,  $\tau \rightarrow \infty$ , and limits

$$k(y) \rightarrow \infty \quad \text{and} \quad \frac{k'(y)}{k(y)} \rightarrow -\infty \quad \text{as} \quad y \rightarrow 0, \quad (4.6)$$

describing infinite density.

**Theorem 4.1.** *The above conditions imply that for gases having energy (4.1), the Riemann problem has a unique global solution which does not admit the vacuum state. Moreover, such gases satisfy all constraints of Section 2 with the exception of (2.6).*

*Proof.* By (4.1) and (2.3), we have

$$p(\tau, S) = \frac{1}{z} k^2(\ln z) f(S), \quad (4.7)$$

where we have set  $z = \tau + 1$ . It is easy to check that the standard thermodynamic constraints, stability of matter, and thermodynamic stability (2.5) are satisfied. The energy condition follows from the first equation in (4.5).

We now check that the vacuum condition (3.11) fails for this equation of state. We calculate

$$p_\tau(\tau, S) = -\frac{k^2(\ln z) - 2k(\ln z)k'(\ln z)}{z^2}f(S) < 0,$$

and, since  $\rho = 1/\tau$ , we write

$$\hat{p}_\rho(\rho, S) = -p_\tau(\tau, S)\tau^2,$$

and so, by (3.3),

$$\begin{aligned} R(\rho, S) &= -\int_1^{1/\rho} \sqrt{-p_\tau(\tau, S)} d\tau \\ &= -\sqrt{f(S)} \int_{\ln 2}^{\ln(1+1/\rho)} \sqrt{k^2(w) - 2k(w)k'(w)} dw, \end{aligned} \quad (4.8)$$

where  $w = \ln z = \ln(1 + \tau)$ . It now follows from (4.4), (4.5) that

$$R(\rho, S) < -\sqrt{f(S)} \int_{\ln 2}^{\ln(1+1/\rho)} k(w) dw \rightarrow -\infty$$

as  $\rho \rightarrow 0$  for all  $S$ , so that (3.11) is never satisfied.

Existence and uniqueness of Riemann solutions with arbitrary data now follows from Smith's medium condition (2.10), see [15]. Smith has an extra assumption, namely, he requires

$$\lim_{\tau \rightarrow 0^+} E(p, \tau) = 0, \quad (4.9)$$

when  $E$  is described as  $E = E(p, \tau)$ . Because our energy is separable, we have

$$E(p, \tau) = p \frac{z \int_{\ln z}^{\infty} k^2(y) dy}{k^2(\ln z)}. \quad (4.10)$$

Since  $z \rightarrow 1$  as  $\tau \rightarrow 0$ , for small  $\tau$ , we write

$$E(p, \tau) = p z \frac{\int_{\ln z}^1 k^2(y) dy + \int_1^{\infty} k^2(y) dy}{k^2(\ln z)}.$$

By our assumptions on  $k$ , (4.9) follows if we show that the limit

$$\frac{\int_{\varepsilon}^1 k^2(y) dy}{k^2(\varepsilon)} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

This in turn follows from (4.6) and l'Hospital's rule.

To check the medium condition (2.10), we use (4.10) to write

$$p(\tau, E) = E \frac{k^2(\ln z)}{z \int_{\ln z}^{\infty} k^2(y) dy},$$

where  $z = \tau + 1$ . The medium condition (2.10) is

$$\frac{2k(\ln z)k'(\ln z) \int_{\ln z}^{\infty} k^2(y) dy - k^2(\ln z) \int_{\ln z}^{\infty} k^2(y) dy + k^4(\ln z)}{(z \int_{\ln z}^{\infty} k^2(y) dy)^2} E < \frac{p^2}{2E},$$

which simplifies to

$$N(z) := \frac{1}{2}k^3(z) + 2k'(z) \int_z^{\infty} k^2(y) dy - k(z) \int_z^{\infty} k^2(y) dy < 0,$$

which must hold for all  $z > 1$ .

By (4.5),  $k(z) \rightarrow 0$  as  $z \rightarrow \infty$ , which in turn implies that  $N(z) \rightarrow 0$  as  $z \rightarrow \infty$ . Furthermore, by (4.4),

$$N'(z) = -\frac{1}{2}k'k^2 + 2k'' \int_z^{\infty} k^2 - k' \int_z^{\infty} k^2 + k^3 > 0,$$

so  $N$  is increasing with limit 0, and thus  $N(z) < 0$  for all  $z$ .  $\square$

We remark that our conditions do not suffice to prove the convexity of  $E(\tau, S)$ , namely the first condition of (2.6),

$$E_{\tau\tau} \cdot E_{SS} \geq E_{\tau S}^2.$$

This condition is easily seen to be implied by the dual assumptions that  $f$  is log-convex,

$$f(S) f''(S) \geq f'^2(S), \quad (4.11)$$

and  $k$  satisfies the condition

$$(k(z) - 2k'(z)) \int_z^{\infty} k^2(y) dy \geq k^3(z). \quad (4.12)$$

for all  $z$ .

## 5 Concrete example

We now present the simplest equation of state which does not allow for a vacuum in the solution of the Riemann problem. Ideal polytropic ( $\gamma$ -law) gases with adiabatic constant  $\gamma > 1$  admit vacuums, while an isothermal gas ( $\gamma = 1$ ) does not, but the isothermal gas does not satisfy the finite energy stability of matter condition. We thus consider equations of state that fall between these two cases. Since (3.12) implies existence of vacuums, this restricts the equation of state to those for which (3.12) fails.

This class of gases we present here are polytropic, satisfying

$$E = c_\tau T,$$

but do not satisfy the ideal gas law  $p\tau = RT$ . For a polytropic gas of the form (4.1), (2.3) implies that we must have

$$f(S) = K_0 e^{S/c_\tau},$$

and this trivially satisfies conditions (4.2), (4.3) and (4.11).

We choose  $k = k(z)$  as simple as possible so that properties (4.4) and (4.5) hold, namely

$$k(z) = z^{-\sigma}, \quad \text{for some } \sigma \in (\tfrac{1}{2}, 1]. \quad (5.1)$$

It is then easy to check that (4.4), (4.5), (4.6) and (4.12) hold.

With these choices, (4.1) becomes

$$E = \frac{K_0 e^{S/c_\tau}}{2\sigma - 1} (\ln(\tau + 1))^{1-2\sigma}, \quad (5.2)$$

and (4.7) becomes

$$p = \frac{K_0 e^{S/c_\tau}}{\tau + 1} (\ln(\tau + 1))^{-2\sigma}. \quad (5.3)$$

It follows from Theorem 4.1 that the Riemann problem has a unique solution without vacuum. In fact, Smith's strong condition holds, namely

$$\frac{\partial}{\partial \tau} E(\tau, p) > 0.$$

This is easily seen by eliminating  $e^{S/c_\tau}$  from (5.2) and (5.3), to get

$$E(\tau, p) = p \frac{(\tau + 1) \ln(\tau + 1)}{2\sigma - 1},$$

and differentiating.

Although we have an abstract proof of existence and uniqueness of Riemann solutions, we find it instructive to prove this directly.

**Lemma 5.1.** *There is a unique solution to the Riemann problem for (1.1) with equation of state (5.2), which does not include the vacuum, for arbitrary Riemann data.*

*Proof.* Using (4.8) and simplifying, we have

$$R(\tau, S) = -\sqrt{K_0} e^{S/2c_\tau} \int_{\ln 2}^{\ln(\tau+1)} w^{-\sigma} \sqrt{1 + \frac{2\sigma}{w}} dw. \quad (5.4)$$

Following [2], we now make convenient changes of variables. First, set

$$\phi := \ln(\tau + 1) > 0, \quad \text{so that} \quad \tau + 1 = e^\phi,$$

and define

$$h := e^{-S/2c_\tau} R(\tau, S) = -\sqrt{K_0} \int_{\ln 2}^{\phi} w^{-\sigma} \sqrt{1 + \frac{2\sigma}{w}} dw.$$

It is clear that  $\rho$ ,  $\tau$ ,  $\phi$  and  $h$  are equivalent coordinates, with  $\phi$  and  $\tau$  decreasing and  $h$  increasing as functions of  $\rho$ , and since  $\sigma \in (\frac{1}{2}, 1]$ , we have

$$\lim_{\phi \rightarrow \infty} h = -\infty \quad \text{and} \quad \lim_{\phi \rightarrow 0} h = \infty. \quad (5.5)$$

Next, define

$$m := \phi^{-\sigma} e^{S/2c_\tau} = (\ln(\tau + 1))^{-\sigma} e^{S/2c_\tau},$$

so that  $(\phi, m)$  can be used in place of  $(\tau, S)$  or  $(\rho, S)$ .

It follows from (5.2) and (5.3) that

$$E = \frac{K_0}{2\sigma - 1} \phi m^2 \quad \text{and} \quad p = K_0 e^{-\phi} m^2, \quad (5.6)$$

while also

$$e^{S/2c_\tau} = m \phi^\sigma \quad \text{and} \quad R = h m \phi^\sigma.$$

The simple wave curves (3.2), (3.4) are described by

$$u_r - u_l = (h_a - h_b) m_a \phi_a^\sigma, \quad \frac{m_b}{m_a} = \frac{\phi_a^\sigma}{\phi_b^\sigma}, \quad (5.7)$$

where the subscripts denote the behind, ahead, right and left states respectively, and  $\phi = \phi(h)$ . For rarefaction waves, the sound speed  $c$  decreases from front to back, so these are characterized by  $h_a > h_b$ . Similarly, by (3.10), a contact discontinuity is described by

$$u_r = u_l, \quad \frac{m_r}{m_l} = e^{(\phi_r - \phi_l)/2}. \quad (5.8)$$

It remains to calculate the shock curves. Using (5.6) in (3.8), we get

$$\frac{K_0}{2\sigma-1}(\phi_b m_b^2 - \phi_a m_a^2) + K_0 \frac{e^{-\phi_b} m_b^2 + e^{-\phi_a} m_a^2}{2} (e^{\phi_b} - e^{\phi_a}) = 0,$$

which becomes

$$\frac{m_b}{m_a} = \sqrt{\frac{\frac{1}{2\sigma-1}\phi_a + \frac{1}{2} - \frac{1}{2}e^{\phi_b-\phi_a}}{\frac{1}{2\sigma-1}\phi_b + \frac{1}{2} - \frac{1}{2}e^{\phi_a-\phi_b}}} =: f(\phi_a, \phi_b). \quad (5.9)$$

Here,  $\phi_a > \phi_b$ , or equivalently  $h_a < h_b$ , since the sound speed  $c$  is greater behind the shock [6, 9]. It is clear that the function  $f$  in (5.9) makes sense only if the function inside the square root is nonnegative. To check that this holds, consider the function

$$q(x, y) := \frac{\frac{1}{2\sigma-1}x + \frac{1}{2} - \frac{1}{2}e^{y-x}}{\frac{1}{2\sigma-1}y + \frac{1}{2} - \frac{1}{2}e^{x-y}},$$

for  $x > y > 0$ . Denote the numerator and denominator of  $q$  by

$$\begin{aligned} qn(x, y) &:= \frac{1}{2\sigma-1}x + \frac{1}{2} - \frac{1}{2}e^{y-x} \quad \text{and} \\ qd(x, y) &:= \frac{1}{2\sigma-1}y + \frac{1}{2} - \frac{1}{2}e^{x-y}, \end{aligned}$$

respectively. It is immediate that for  $x > 0$ ,

$$qn(x, x) > 0, \quad qn(x, 0) > 0, \quad qd(x, x) > 0, \quad qd(x, 0) < 0.$$

Since  $qd(x, y)$  is strictly increasing with respect to  $y$ , for each  $x$ ,  $qd(x, y) = 0$  has a unique solution  $0 < y = \varphi(x) < x$ . On the other hand,  $qn(x, y)$  is strictly decreasing with respect to  $y$ , so  $qn(x, y) > 0$  for  $x > y > 0$ . Hence, for fixed  $x$ ,

$$q(x, y) > 0 \quad \text{as long as} \quad \varphi(x) < y < x,$$

while also

$$\lim_{y \rightarrow \varphi(x)^+} q(x, y) = +\infty.$$

It follows that, for fixed  $\phi_a$ ,  $0 < \varphi(\phi_a) < \phi_a$  and the shock curve is parameterized by  $\phi_b \in (\varphi(\phi_a), \phi_a)$ , with

$$\lim_{\phi_b \rightarrow \varphi(\phi_a)^+} f(\phi_a, \phi_b) = +\infty. \quad (5.10)$$

Moreover,  $f(\phi_a, \phi_b)$  increases with respect to  $\phi_a$  and decreases with respect to  $\phi_b$ . Since  $qd(x, y) = qn(y, x)$ , we also have

$$f(\phi_a, \phi_b) = \frac{1}{f(\phi_b, \phi_a)}.$$

Next, in these coordinates, (3.9) is

$$\begin{aligned} [u] &= -\sqrt{K_0} \sqrt{(e^{-\phi_b} m_b^2 - e^{-\phi_a} m_a^2)(e^{\phi_a} - e^{\phi_b})} \\ &= -\sqrt{K_0} \sqrt{(e^{\phi_a - \phi_b} \frac{m_b^2}{m_a^2} - 1)(1 - e^{\phi_b - \phi_a}) m_a}. \end{aligned} \quad (5.11)$$

Defining

$$g(\phi_a, \phi_b) := -\sqrt{K_0} \sqrt{(e^{\phi_a - \phi_b} f^2(\phi_a, \phi_b) - 1)(1 - e^{\phi_b - \phi_a})}, \quad (5.12)$$

it is easy to check that

$$g(\phi_a, \phi_b) m_a = g(\phi_b, \phi_a) m_b,$$

provided (5.9) holds, and  $g(\phi_a, \phi_b)$  decreases with respect to  $\phi_a$  and increases with respect to  $\phi_b$ . By (5.10), since  $\phi_a > \phi_b$ ,

$$\lim_{\phi_b \rightarrow \varphi(\phi_a)} g(\phi_a, \phi_b) = -\infty. \quad (5.13)$$

To express the composite wave curves, we define

$$\begin{aligned} G(h_a, h_b) &:= \begin{cases} g(\phi_a, \phi_b), & h_a < h_b < h(\varphi(\phi_a)), \\ (h_a - h_b) \phi_a^\sigma, & h_a \geq h_b, \end{cases} \quad \text{and} \\ F(h_a, h_b) &:= \begin{cases} f(\phi_a, \phi_b), & h_a < h_b < h(\varphi(\phi_a)), \\ \phi_a^\sigma / \phi_b^\sigma, & h_a \geq h_b, \end{cases} \end{aligned}$$

with  $\phi = \phi(h)$ , and  $\varphi(x)$  is defined by  $qd(x, \varphi(x)) = 0$ . Both  $F(h_a, h_b)$  and  $G(h_a, h_b)$  are defined on the region  $\{h_b < \varphi(h_a)\}$ , with  $F$  increasing and  $G$  decreasing in  $h_b$  for fixed  $h_a$ . Moreover, by (5.5), we have the limits

$$\lim_{h_b \rightarrow -\infty} F(h_a, h_b) = 0 \quad \text{and} \quad \lim_{h_b \rightarrow -\infty} G(h_a, h_b) = +\infty, \quad (5.14)$$

for any  $h_a$  fixed, and by (5.10) and (5.13),

$$\lim_{h_b \rightarrow h(\varphi(\phi_a))} F(h_a, h_b) = +\infty \quad \text{and} \quad \lim_{h_b \rightarrow h(\varphi(\phi_a))} G(h_a, h_b) = -\infty. \quad (5.15)$$

Combining (5.7) and (5.9), (5.11), we describe the composite 1- and 3-wave curves by

$$u_r - u_l = G(h_a, h_b)m_a, \quad \frac{m_b}{m_a} = F(h_a, h_b), \quad (5.16)$$

while the 2-waves are given by (5.8), namely

$$u_r = u_l, \quad \frac{m_r}{m_l} = e^{\frac{\phi_r - \phi_l}{2}}.$$

We now solve the Riemann problem by resolving the intermediate states. We use subscripts  $L$ , 1, 2 and  $R$  to denote the left, intermediate, and right states, respectively. We set

$$A := \frac{u_R - u_L}{m_L} \quad \text{and} \quad B := \frac{m_R}{m_L},$$

and use (5.16) and (5.8) to describe the waves, and eliminate  $u$ , to get the equations

$$G(h_L, h_1) + B G(h_R, h_2) = A, \quad \text{and} \quad (5.17)$$

$$F(h_L, h_1) e^{-\phi_1/2} = B F(h_R, h_2) e^{-\phi_2/2}, \quad (5.18)$$

and we must solve for  $h_1$  and  $h_2$ .

For any fixed  $h_0$ , the function  $F(h_0, h) e^{-\phi(h)/2}$  is increasing in  $h$ , and by (5.14), (5.15), it has range  $(0, \infty)$ . Thus there exists a unique (increasing and  $C^2$ ) function  $\Theta$  of  $h_2$ , such that

$$h_1 = \Theta(h_2; B, h_L, h_R)$$

if and only if (5.18) holds. Substituting into (5.17) gives the equation

$$G(h_L, \Theta(B, h_L, h_R, h_2)) + B G(h_R, h_2) = A,$$

and we must solve for  $h_2$ . Since  $G(h_0, h)$  is decreasing with  $h$ , and by (5.14), (5.15) has range  $(-\infty, \infty)$ , there is a unique solution  $h_2 < \varphi(\phi_R)$  of this equation. Note that we also have  $h_1 < \varphi(\phi_L)$ . Finally, we use (5.16) to fully resolve the intermediate states, and the construction of the Riemann solution is complete.  $\square$



## References

- [1] A. Bressan, *Hyperbolic Systems of Conservation laws: The One-dimensional Cauchy Problem*, Oxford Lecture Ser. Math. Appl. 20, Oxford Univ. Press, Oxford, 2000.
- [2] Geng Chen, *Formation of singularity and smooth wave propagation for the compressible Euler equations*, to appear in J. Hyp. Diff. Eq..
- [3] G. Chen, E. Endres and K. H. Jenssen, *Pairwise wave interactions in ideal polytropic gases*, to appear in Arch. Rat. Mech. Anal.
- [4] G. Chen and R. Young, *Smooth solutions and singularity formation for the inhomogeneous nonlinear wave equation*, J. Diff. Eq., **252**(2012), 2580–2595.
- [5] G. Chen and R. Young, *Shock formation and exact solutions for the compressible Euler equations with piecewise smooth entropy*, submitted.
- [6] R. Courant and K. O. Friedrichs, *Supersonic Flow and Shock Waves*, Wiley-Interscience, New York, 1948.
- [7] C. M. Dafermos, *Hyperbolic Conservation laws in Continuum Physics*, Springer-Verlag, Heidelberg, 2000.
- [8] J. Glimm, *Solutions in the large for nonlinear hyperbolic systems of equations*, Comm. Pure Appl. Math., **18**(1965), 697–715.
- [9] P. D. Lax, *Hyperbolic systems of conservation laws, II*, Comm. Pure Appl. Math., **10**(1957), 537–566.
- [10] T.-T. Li and Y. Zhao, *Vacuum problems for the systems of one-dimensional isentropic flow (in Chinese)*, Chinese Quarterly Journal of Math., **1**(1986), 41–46.
- [11] E. Lieb, *The stability of matter*, Rev. Mod. Phys. **48**(1976), 553.
- [12] T.-P. Liu and J. Smoller, *On the vacuum state for the isentropic gas dynamics equations*, Adv. in Appl. Math., **1**(1980), 345–359.
- [13] R. Menikoff, B. J. Plohr, *The Riemann problem for fluid flow of real materials*, Reviews of Modern Physics, **61**:1(1989), 75–130.
- [14] T. Nishida, *Global solutions for an initial boundary value problem of a quasilinear hyperbolic system*, Proc. Japan Acad., **44**(1968), 642–646.

- [15] R. Smith, *The Riemann problem in gas dynamics*, Trans. Am. Math. Soc., **249**(1979), 1–50.
- [16] J. Smoller, *Shock waves and reaction-diffusion equations*, Springer-Verlag, New York, 1982.
- [17] B. Temple, *Solutions in the large for the nonlinear hyperbolic conservation laws of gas dynamics*, J. Diff. Eqns., **41**(1981), 96–161.
- [18] B. Temple and R. Young, *The large time stability of sound waves*, Comm. Math. Phys., **179**(1996), 417–466.
- [19] Blake Temple and Robin Young, *A paradigm for time-periodic sound wave propagation in the compressible Euler equations*, Methods and Appls of Analysis, **16**(2009), 341–364.
- [20] R. Young, *The  $p$ -system II: The vacuum*, Evolution Equations Banach Center Publications **60**(2003), 237–252.